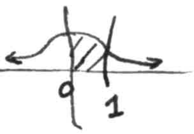
Integration Review Worksheet  Math 1131  Class #39

Name: Solutions  Section: ______

## Net Area and Integrals



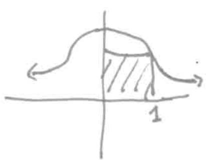

1. The integral $\int_a^b e^{-x^2}\,dx$ is important in statistics,[1] but it is infamously hard to compute. Many statistics textbooks include a table which lists the value of the integral for different values of $a$ and $b$. We will use Reimann Sums to generate one of these approximations.

(a) Express the integral $\displaystyle\int_0^1 e^{-x^2}\,dx$ as the limit of its Right Reimann Sums.

$$\int_0^1 e^{-x^2}dx = \lim_{n\to\infty}\sum_{i=1}^{n} e^{-(x_i)^2}\cdot\Delta x$$

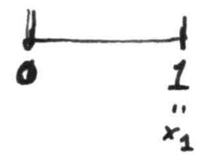
where $\Delta x = \frac{1-0}{n}$ and $x_i = a + i\cdot\Delta x$

(b) Approximate $\displaystyle\int_0^1 e^{-x^2}\,dx$ using Right Sums and $n = 1$. Use a calculator to simplify.




$R_1 = f(x_1)\cdot\Delta x$  $\Delta x = \frac{1-0}{1}$  $x_1 = 1$

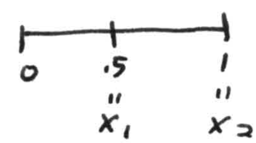
$= e^{-(1)^2}\cdot 1 \approx .36788$



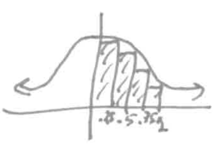

(c) Approximate $\displaystyle\int_0^1 e^{-x^2}\,dx$ using Right Sums and $n = 2$. Use a calculator to simplify.


$R_2 = f(x_1)\Delta x + f(x_2)\Delta x$  $\Delta x = \frac{1-0}{2} = .5$

$= e^{-(.5)^2}(.5) + e^{-(1)^2}(.5) \approx .57334$




(d) Approximate $\displaystyle\int_0^1 e^{-x^2}\,dx$ using Right Sums and $n = 4$. Use a calculator to simplify.

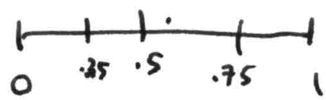


$R_4 = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$

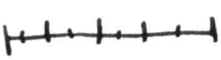
$= e^{-(.25)^2}(.25) + e^{-(.5)^2}(.25) + e^{-(.75)^2}(.25) + e^{-(1)^2}(.25)$

$\approx .66397$

$\Delta x = \frac{1-0}{4} = .25$




$\Delta x = \frac{1-0}{8} = .125$



(e) Approximate $\displaystyle\int_0^1 e^{-x^2}\,dx$ using Right Sums and $n = 8$. Use a calculator to simplify.

$R_8 = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x + f(x_7)\Delta x + f(x_8)\Delta x$

$= (.125)\left[e^{-(.125)^2} + e^{-(.25)^2} + e^{-(.375)^2} + e^{-(.5)^2} + e^{-(.625)^2} + e^{-(.75)^2} + e^{-(.875)^2} + e^{-(1)^2}\right]$

$\approx .70636$

(f) How do these compare to the correct value of $\displaystyle\int_0^1 e^{-x^2}\,dx = .7468241\ldots$ ?

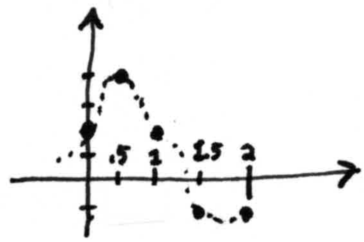
they get closer as n gets bigger.

[1]This and other similar integrals are needed to compute the probability of events that follow a normal distribution. See, for example, http://en.wikipedia.org/wiki/Standard_normal_distribution#Cumulative_distribution.

Name: ______________________________ Section: ______________________________

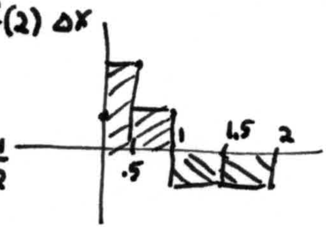
2. A certain function is defined using the table below.

| $x$ | 0 | 0.5 | 1 | 1.5 | 2 |
|---|---|---|---|---|---|
| $f(x)$ | 2 | 4 | 2 | -1 | -1 |




(a) Approximate $\int_0^2 f(x)\,dx$ using right sums and four rectangles.

$$\approx R_4 = f(0.5)\,\Delta x + f(1)\Delta x + f(1.5)\Delta x + f(2)\,\Delta x$$




$$= 4\cdot\tfrac{1}{2} + 2\cdot\tfrac{1}{2} + (-1)\cdot\tfrac{1}{2} + (-1)\cdot\tfrac{1}{2}$$

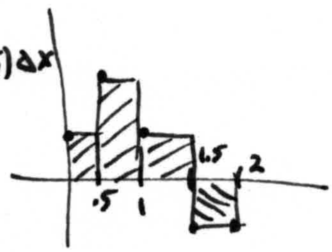
$$= 2 + 1 - 1$$

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}$$

$$= 2$$

(b) Approximate $\int_0^2 f(x)\,dx$ using left sums and four rectangles.

$$\approx L_4 = f(0)\Delta x + f(0.5)\Delta x + f(1)\Delta x + f(1.5)\Delta x$$



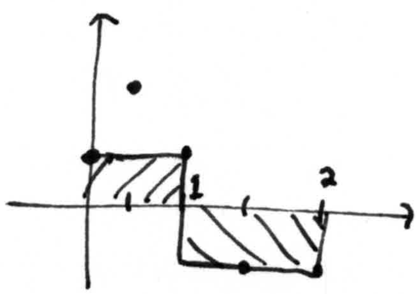

$$= 2\cdot\tfrac{1}{2} + 4\cdot\tfrac{1}{2} + 2\cdot\tfrac{1}{2} + (-1)\cdot\tfrac{1}{2}$$

$$= 1 + 2 + 1 - 0.5$$

$$= 3.5$$

(c) Approximate $\int_0^2 f(x)\,dx$ using right sums and two rectangles.

$$R_2 = f(1)\cdot\Delta x + f(2)\cdot\Delta x$$




$$= 2\cdot 1 + (-1)\cdot 1$$

$$= 2 - 1$$

$$= 1$$

$$\Delta x = \frac{2-0}{2} = 1$$

Name: ______________________ Section: ______________________

2. Compute the following sums:

(a) $\sum_{k=1}^{3} 2^i - 1$ $= 2^1 - 1 + 2^2 - 1 + 2^3 - 1$

$= 1 + 3 + 7$

$= 11$

(b) $\sum_{i=2}^{6} 2i + 1$ $= 2\cdot2+1 + 2\cdot3+1 + 2\cdot4+1 + 2\cdot5+1 + 2\cdot6+1$

$= 5 + 7 + 9 + 11 + 13$

$= 45$

(c) $\sum_{j=0}^{4} j^2 - 1$ $= 0^2-1 + 1^2-1 + 2^2-1 + 3^2-1 + 4^2-1$

$= -1 + 0 + 3 + 8 + 15$

$= 25$

3. Express $\int_1^4 x^2 + 1\, dx$ as a limit of Right Reimann sums.

$\Delta x = \frac{4-1}{n} = \frac{3}{n}$

$x_i = 1 + i\cdot\Delta x$

$= 1 + i\cdot\frac{3}{n}$

$\int_1^4 x^2+1\, dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n}((x_i)^2+1)\Delta x\right]$

$= \lim_{n\to\infty}\left[\sum_{i=1}^{n}(1+\frac{3i}{n})\cdot\frac{3}{n}\right]$

4. What is the **indefinite** integral defined by the following Reimann sum?

$$\lim_{n\to\infty}\sum_{i=1}^{n}\ln(x_i)\cdot\Delta x$$

$= \int \ln(x)\, dx$

5. What is the **definite** integral defined by the following Reimann sum?

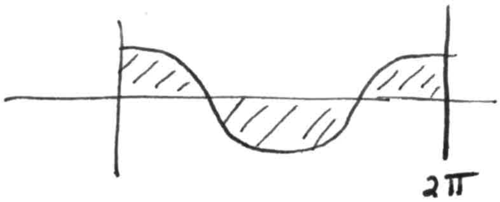
$$\lim_{n\to\infty}\sum_{i=1}^{n}2\cdot\sin\left(\underbrace{2+i\cdot\frac{5-2}{n}}_{x_i}\right)\cdot\underbrace{\frac{5-2}{n}}_{\Delta x}$$

$= \int_2^5 2\cdot\sin(x)\, dx$

Name: ______________________ Section: ______________________

2. What is the graphical meaning of $\int_0^{2\pi} \cos(x)\,dx$? Compute this area geometrically.

the net area



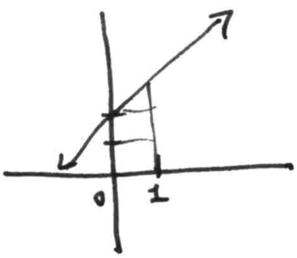

These areas cancel

so

integral = net area = 0

3. What is the graphical meaning of $\int_0^{1} x + 2\,dx$? Compute this area geometrically.

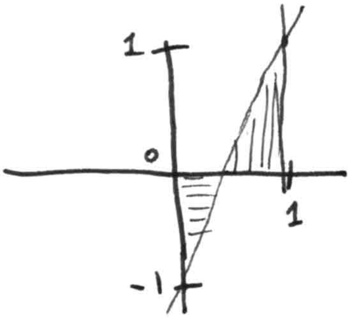
the net area below




net area = $1 + 1 + \frac{1}{2} = \frac{5}{2}$

4. What is the graphical meaning of $\int_0^{1} 2x - 1\,dx$? Compute this area geometrically.

the net area $= \left(\frac{-1}{4}\right) + \left(\frac{1}{4}\right) = 0$

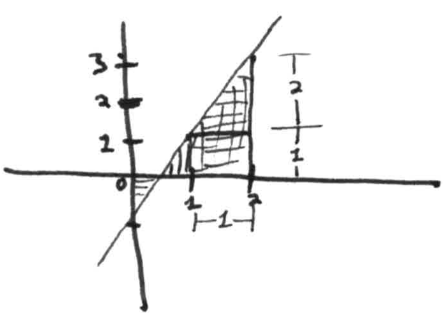



5. What is the graphical meaning of $\int_0^{2} 2x - 1\,dx$? Compute this area geometrically.

the net area

net area $= \left(\frac{-1}{4}\right) + \left(\frac{1}{4}\right) + 1 + \frac{1}{2}(2)(1)$

$= 1 + 1 = 2$

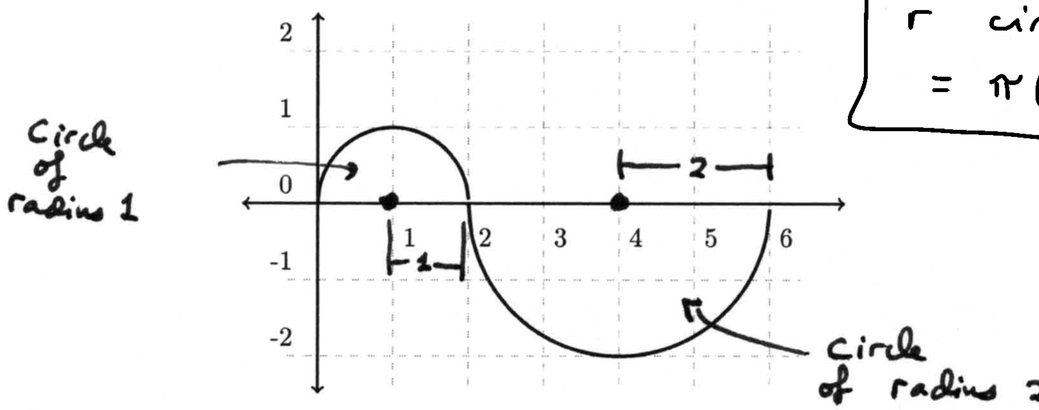


Integration Review Worksheet — Math 1131 — Class #39

Name: ____________________ Section: ____________________

area of radius r circle $= \pi (r)^2$

11. Suppose that the function $f(x)$ is given by the following graph.




(a) Compute $\int_0^1 f(x)\,dx$ $= \frac{1}{4}\left(\text{area of unit circle}\right) = \boxed{\frac{1}{4}\cdot\pi}$

(b) Compute $\int_0^2 f(x)\,dx$ $= \frac{1}{2}\left(\text{area of unit circle}\right) = \boxed{\frac{1}{2}\cdot\pi}$

(c) Compute $\int_0^4 f(x)\,dx$ $= \frac{1}{2}\left(\text{area of unit circle}\right) - \frac{1}{4}\left(\text{area of radius 2 circle}\right) = \frac{1}{2}\pi - \frac{1}{4}\cdot(\pi\cdot 2^2)$

$= \frac{\pi}{2} - \frac{4\pi}{4} = \frac{\pi}{2} - \pi = \frac{-\pi}{2}$

(d) Compute $\int_0^6 f(x)\,dx$ $= \frac{1}{2}(\pi) - \frac{1}{2}(4\pi)$

$= \frac{\pi}{2} - 2\pi = \boxed{\frac{-3\pi}{2}}$

(e) Compute $\int_1^4 f(x)\,dx$

$= \frac{1}{4}(\pi) - \frac{1}{4}(4\pi) = \frac{\pi}{4} - \pi = \boxed{\frac{-3\pi}{4}}$

12. Compute $\int_{-1}^0 \sqrt{1-x^2}\,dx$ geometrically (top half of unit circle)

unit circle:

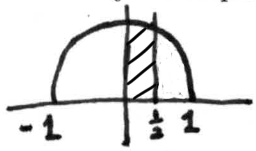
$y^2 + x^2 = 1$

$\Rightarrow y = \pm\sqrt{1-x^2}$


area $= \frac{1}{4}\left(\text{area of unit circle}\right)$

$\boxed{= \frac{1}{4}\pi}$

13. Can you compute the integral $\int_0^{1/2} \sqrt{1-x^2}\,dx$ geometrically? Why or why not?




No. there isn't any symmetry in this picture.

Name: ______________________ Section: ______________________

6. Answer the following **Yes or No**

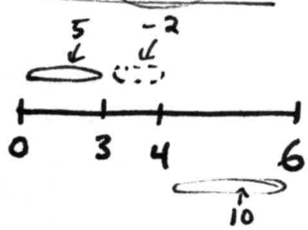
   (a) Can you distribute integrals across sums?
   i.e. does $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$? **(Yes)** No

   (b) Can you pull constants through integrals?
   i.e. does $\int_a^b (c\cdot f(x))\,dx = c\cdot\left(\int_a^b f(x)\,dx\right)$? **(Yes)** No

   (c) Can you distribute integrals across differences?
   i.e. does $\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$? **(Yes)** No

   (d) Can you distribute integrals across products or fractions? Yes **(No!)**

limits are backward → $\int_3^4 f(x)\,dx = -2$

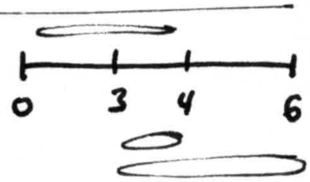


7. Suppose that $\int_0^3 f(x)\,dx = 5$, $\int_4^3 f(x)\,dx = 2$, and $\int_4^6 f(x)\,dx = 10$. Find

   (a) $\int_0^4 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^4 f(x)\,dx = 5 + \left(-\int_4^3 f(x)\,dx\right) = 5 - 2 = 3$

   (b) $\int_3^6 f(x)\,dx = \int_3^4 f(x)\,dx + \int_4^6 f(x)\,dx = (-2) + 10 = 8$

   (c) $\int_0^6 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^4 f(x)\,dx + \int_4^6 f(x)\,dx = 5 - 2 + 10 = 13$



8. Suppose that $\int_0^4 f(x)\,dx = 5$, $\int_3^4 f(x)\,dx = -2$, and $\int_3^6 f(x)\,dx = 10$. Find

   (a) $\int_0^3 f(x)\,dx = \int_0^4 f(x)\,dx - \int_3^4 f(x)\,dx = 5 - (-2) = 7$

   (b) $\int_4^6 f(x)\,dx = \int_3^6 f(x)\,dx - \int_3^4 f(x)\,dx = 10 - (-2) = 12$

   (c) $\int_0^6 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^6 f(x)\,dx = 7 + 10 = 17$

   (7 ← part (a))

Name: ______________________ Section: ______________________

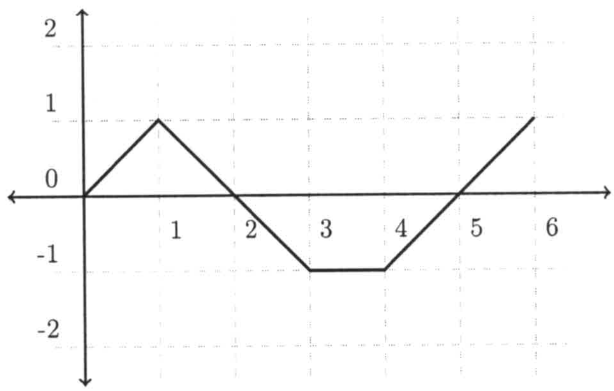
## The Fundamental Theorems

1. Compute $\frac{d}{dx}\left[\int_5^x t^2 + 1\,dt\right]$. $= x^2+1$

2. Compute $\frac{d}{dx}\left[\int_1^x \sin(5t)\,dt\right]$. $= \sin(5x)$

3. Compute $\frac{d}{dx}\left[\int_{-3}^x \sin(\cos(e^t))\,dt\right]$. $= \sin(\cos(e^x))$

4. Suppose that the function $f(x)$ is given by the following graph.




Let $A(x) = \int_0^x f(t)\,dt$. Compute the following

$A'(x) = \frac{d}{dx}\left[\int_0^x f(t)\,dt\right] = f(x)$

(a) $A(1) = \frac{1}{2}$

(b) $A(2) = 1$

(c) $A(4) = -\frac{1}{2}$

(d) $A'(1) = 1$

(e) $A'(2) = 0$

(f) $A'(4) = -1$

Name: ____________________ Section: ____________________

## Computing Integrals Quickly

1. $\int [\sin(x) + e^x - 2\cos(x) + 6x^2]\, dx$

$= \int \sin(x)\,dx + \int e^x\,dx - 2\int \cos(x)\,dx + 6\int x^2\,dx$

$= -\cos(x) + e^x - 2\cdot\sin(x) + 6\cdot\frac{x^3}{3} + C$

$= -\cos(x) + e^x - 2\sin(x) + 2x^3 + C$

2. $\int_0^1 \frac{1}{x^2+1}\,dx = \tan^{-1}(x)\Big]_0^1 = \tan^{-1}(1) - \tan^{-1}(0)$

$\tan^{-1}(1) = x \Leftrightarrow 1 = \tan(x) \Leftrightarrow x = \frac{\pi}{4}$

$\tan^{-1}(0) = x \Leftrightarrow 0 = \tan(x) \Leftrightarrow x = 0$

$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$

3. $\int \frac{x^2+x+1}{x}\,dx = \int \frac{x^2}{x}\,dx + \int \frac{x}{x}\,dx + \int \frac{1}{x}\,dx$

$= \int x\,dx + \int 1\,dx + \int \frac{1}{x}\,dx$

$= \frac{x^2}{2} + x + \ln|x| + C$

4. $\int_{-1}^1 (x^2+3)(x-1)\,dx = \int_{-1}^1 x^3 + 3x - x^2 - 3\;dx$

$= \frac{x^4}{4} + \frac{3x^2}{2} - \frac{x^3}{3} - 3x\Big]_{-1}^1 = \left(\frac{1}{4} + \frac{3}{2} - \frac{1}{3} - 3\right) - \left(\frac{1}{4} + \frac{3}{2} + \frac{-1}{3} + (-3)\right)$

$= \frac{1}{4} - \frac{1}{4} + \frac{3}{2} - \frac{3}{2} - \frac{1}{3} - \frac{1}{3} - 3 - 3$

$= -\frac{2}{3} - 6$

5. $\int \frac{x^{3/2} + \sqrt{x} + 1}{\sqrt{x}}\,dx$

$= \int \frac{x^{\frac{3}{2}}}{\sqrt{x}}\,dx + \int \frac{\sqrt{x}}{\sqrt{x}}\,dx + \int \frac{1}{\sqrt{x}}\,dx$

$= \int \frac{x^{\frac{3}{2}}}{x^{\frac{1}{2}}}\,dx + \int 1\,dx + \int x^{-\frac{1}{2}}\,dx$

$= \int x^{(\frac{3}{2}-\frac{1}{2})}\,dx + \int 1\,dx + \int x^{-\frac{1}{2}}\,dx$

$= \int x\,dx + x + \frac{x^{\frac{1}{2}}}{\frac{1}{2}}$

$= \frac{x^2}{2} + x + 2\sqrt{x} + C$

Name: ________________ Section: ________________

6. $\int x\sqrt{x^2+3}\,dx$

$u = x^2+3$, $\frac{du}{dx} = 2x$, $du = 2x\,dx$, $\frac{du}{2} = x\,dx$

$= \int \sqrt{u}\,\frac{du}{2} = \frac{1}{2}\int u^{\frac{1}{2}}\,du = \frac{1}{2}\cdot\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{u^{\frac{3}{2}}}{3} + C = \frac{(x^2+3)^{\frac{3}{2}}}{3} + C$

7. $\int \frac{1}{x+1}\,dx$

$u = x+1$, $\frac{du}{dx} = 1$, $du = dx$

$= \int \frac{1}{u}\,du = \ln|u| + c = \ln|x+1| + C$

8. $\int \sin(5x) + 1 \;\, dx$

$= \int \sin(5x)\,dx + \int 1\,dx$

$u = 5x$, $\frac{du}{dx} = 5$, $\frac{du}{5} = dx$

$= \int \sin(u)\cdot\frac{du}{5} + x = \frac{\cos(u)}{5} + x + C$

$= \frac{\cos(5x)}{5} + x + C$

9. $\int \cos(x)\cdot e^{\sin(x)}\;\, dx$

$u = \sin(x)$, $\frac{du}{dx} = \cos(x)$, $du = \cos(x)\,dx$

$= \int e^u\,du = e^u + c$

$= e^{\cos(x)} + C$

10. $\int \left[\frac{1}{x^2+1} + \frac{1}{x+1}\right]\,dx = \int \frac{1}{x^2+1}\,dx + \int \frac{1}{x+1}\,dx$

$u = x+1$, $\frac{du}{dx} = 1$, $du = dx$

$= \tan^{-1}(x) + \int \frac{1}{u}\,du$

$= \tan^{-1}(x) + \ln|u| + c$

$= \tan^{-1}(x) + \ln|x+1| + C$

Name: ______________________ Section: ______________________

11. $\int_e^{e^2} \frac{1}{x\ln(x)}\,dx$ $du$

| | |
|---|---|
| $u = \ln(x)$ | $x = e \Rightarrow u = \ln(e) = 1$ |
| $\frac{du}{dx} = \frac{1}{x}$ | $x = e^2 \Rightarrow u = \ln(e^2) = 2$ |
| $du = \frac{1}{x}\,dx$ | |

$$= \int_1^2 \frac{1}{u}\,du$$

$$= \ln|u|\Big]_1^2$$

$$= \ln(2) - \ln(1) = \ln(2)$$

12. $\int_{\pi/4}^{\pi/2} \frac{\cos(x)}{\sin^2(x)}\,dx$

| | |
|---|---|
| $u = \sin(x)$ | $x = \frac{\pi}{4} \Rightarrow u = \sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$ |
| $\frac{du}{dx} = \cos(x)$ | $x = \frac{\pi}{2} \Rightarrow u = \sin(\frac{\pi}{2}) = 1$ |
| $du = \cos(x)\,dx$ | |

$$= \int_{\frac{1}{\sqrt{2}}}^1 \frac{1}{u^2}\,du \qquad \left(\frac{1}{u^2} = u^{-2}\right) \qquad \left(\frac{u^{-1}}{-1} = \frac{-1}{u}\right)$$

$$= \frac{u^{-1}}{-1}\Big]_{\frac{1}{\sqrt{2}}}^1 = \frac{-1}{1} - \frac{-1}{\left(\frac{1}{\sqrt{2}}\right)} = -1 + \sqrt{2}$$